Analogs and a Variation of Erdős's Distinct Distances Problem and Unit Distance Problem for Non-Archimedean Spaces

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Abstract

We explore analogs of Erdős's 1946 distinct distance and unit distance problems in d-dimensional p-adic spaces using the ℓ_∞ distance metric. For these problems, we determine explicit bounds and demonstrate their tightness with simple constructions. Additionally, we study a variation of the distinct distance problem, focusing on maximizing rather than minimizing the number of distinct distances in vector spaces with a non-Archimedean translation-invariant metric. We show that $N \geq 3$ pairwise distinct points in such spaces determine at most $N-1$ distinct distances, with equality in $\mathbb{Q}_p^d$ (in contrast to $\sim N^2$ distinct distances in $\mathbb{R}^n$).

1 Introduction

In 1946, Erdős posed the distinct distance problem, a combinatorial geometry question that seeks to determine the minimal number of distinct distances over all sets of n pairwise distinct points in the Euclidean plane. Erdős proved an upper bound of $O\left(\frac{n}{\sqrt{\log(n)}}\right)$ and conjectured that this is tight [1]. In 2015, Guth and Katz established the nearly tight lower bound of $\Omega\left(\frac{n}{\log(n)}\right)$ [2].

The unit distance problem was also posed by Erdős in 1946 and seeks to determine the maximum number of pairs of points at a fixed distance (e.g., 1) in $\mathbb{R}^2$ given n points. Erdős's construction in his original paper provides the best known lower bound of $O\left(n^{1+\frac{c}{\log(\log n)}}\right)$ [1]. The best known upper bound of $O(n^{4/3})$ was proven by Spencer and Szemerédi using a variation of the Szemerédi-Trotter theorem that bounds incidences between points and unit circles instead of points and lines [3].

In this paper, we present new results on analogs of these Erdős distance problems in d-dimensional p-adic spaces and, more broadly, in vector spaces equipped with non-Archimedean translation-invariant metrics. Our main contributions address analogs of the classical distinct distance and unit distance problems introduced by Erdős in the real setting, as well as a related question on maximizing the number of distinct distances. All our results use the ℓ_∞ distance metric, defined as follows:

Definition 1.1 (ℓ_∞ distance)**.** Let $\mathbb{K}$ be a metric space. For two points $x, y \in \mathbb{K}^d$, which can be expressed as

$$x = (x_1, x_2, \ldots, x_d), \quad y = (y_1, y_2, \ldots, y_d),$$

the ℓ_∞ distance is defined as

$$D(x, y) = \max_{1 \le i \le d} |x_i - y_i|_K.$$

Using this metric, we establish explicit bounds on distinct and unit distances in d-dimensional p-adic spaces, highlighting how non-Archimedean norms constrain point configurations. In addition, our work connects to related research exploring distance-type incidence problems in p-adic spaces, finite fields, and other non-Euclidean settings [4, 5, 6].

Theorem 1.1. *The minimal number of distinct ℓ_∞ distances over all sets of n pairwise distinct points in $\mathbb{Q}_p^d$ is $\lceil \log_p n^{1/d} \rceil$.*

Theorem 1.2. *The maximum number of unit ℓ_∞ distances over all sets of n pairwise distinct points in $\mathbb{Q}_p^d$ is*

$$\frac{1}{2}(n^2 - n\alpha - \alpha\beta - \beta),$$

where α and β are the unique non-negative integers such that $\beta < p^d$ and $n = \alpha p^d + \beta$.

In Section 2, we review fundamental definitions of p-adic spaces, their norms, and non-Archimedean absolute values. We then prove theorems 1.1 and 1.2 in Sections 3 and 4, respectively. In Section 5, we consider a variation of the distinct distances problem for vector spaces equipped with a non-Archimedean translation-invariant metric. Rather than seeking the *minimal* number of distinct distances determined by n points, we investigate the *maximum* possible. In $\mathbb{R}^n$, the answer is straightforward; however, in non-Archimedean spaces where all triangles are isosceles, the solution becomes nontrivial.

Theorem 1.3. *A set of $n \geq 3$ pairwise distinct points in a vector space equipped with a non-Archimedean translation-invariant metric determines at most $n-1$ ℓ_∞ distances.*

We conclude Section 5 by proving that this upper bound is attained in $\mathbb{Q}_p^d$ (Corollary 1.4).

Corollary 1.4. *The maximum number of distinct ℓ_∞ distances for n pairwise distinct points in $\mathbb{Q}_p^d$ is $n-1$.*

2 Background

In this section, we review fundamental definitions and relations about the p-adic numbers used in this paper. Details can be found in [7].

Definition 2.1. A *p-adic number* is a formal series of the form

$$a = \sum_{i=u}^{+\infty} a_i p^i,$$

where p is prime, $u \in \mathbb{Z}$, $a_u \neq 0$, and $a_i \in \{0, 1, \ldots, p-1\}$.

The set of all such numbers for a fixed p is the set of p-adic numbers, denoted by $\mathbb{Q}_p$.

Definition 2.2. Let x be a non-zero integer. The *p-adic valuation* of x, denoted $v_p(x)$, is the unique positive integer satisfying

$$x = p^{v_p(x)} x',$$

such that $p \nmid x'$. Additionally, we define $v_p(0) = +\infty$.

Notice that the p-adic valuation is equivalent to its order when expanded as a power series. Also, we can extend this definition to rational numbers by writing

$$v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b).$$

Definition 2.3. The *p-adic norm* of a p-adic number a is given by

$$|a|_p = \begin{cases} p^{-v_p(a)} & a \neq 0 \\ 0 & a = 0. \end{cases}$$

The p-adic norm induces the p-adic metric, denoted as d_p, by $d_p(a,b) = |a-b|_p$.

Definition 2.4. A metric $d\colon \mathbb{K} \times \mathbb{K} \to \mathbb{R}$ is *non-Archimedean* if for all points $a, b, c \in \mathbb{K}$,

$$d(a, b) \le \max\{d(a, c), d(c, b)\}.$$

We call a set equipped with a non-Archimedean metric an ultrametric space.

Definition 2.5. A metric $d\colon \mathbb{K} \times \mathbb{K} \to \mathbb{R}$ is *translation-invariant* if for all points $a, b, c \in \mathbb{K}$,

$$d(a, b) = d(a - c, b - c).$$

Importantly, the p-adic metric is non-Archimedean and translation-invariant. To see this, first observe that for any $a, b, c \in \mathbb{Q}_p$,

$$d_p(a - c, b - c) = \big|(a - c) - (b - c)\big|_p = |a - b|_p = d_p(a, b),$$

so the metric is translation-invariant. Next, for any $x, y \in \mathbb{Q}_p$, the valuation satisfies

$$v_p(x + y) \ \ge \ \min\{v_p(x),\, v_p(y)\},$$

hence

$$|x + y|_p = p^{-\,v_p(x+y)} \le p^{-\min\{v_p(x), v_p(y)\}} = \max\{p^{-v_p(x)},\, p^{-v_p(y)}\} = \max\{|x|_p,\, |y|_p\},$$

which is equivalent to the inequality given by Definition 2.4.

Proposition 2.1. *Let $\mathbb{K}$ be an ultrametric space, and let $x, y \in \mathbb{K}$. If $|x| \ne |y|$, then $|x + y| = \max(|x|, |y|)$.*

Proof. Without loss of generality, we assume that $|x| > |y|$. By Definition 2.4, we know that $|x + y| \le |x|$. However, since

$$x = (x + y) + (-y),$$

we can write

$$|x| = |(x + y) + (-y)| \le \max\{|x + y|.|y|\}.$$

By our initial assumption, we must have $|x| \le |x + y|$. Hence, we have

$$|x + y| \le |x| \le |x + y|,$$

so $|x + y| = |x|$ as desired. ☐

We can apply Proposition 2.1 to the p-adics to show that any set of three points in $\mathbb{Q}_p$ form an isosceles triangle under the p-adic metric.

Lemma 2.2. *For any three distinct points in $\mathbb{Q}_p$, at least one of them is equidistant from the other two.*

Proof. If $d_p(a, b) = d_p(b, c)$, then the result is clearly true. Otherwise, writing $(a - b) + (b - c) = (a - c)$ and applying Lemma 2.1 gives:

$$|(a - b) + (b - c)| = |a - c| = \max\{|a - b|, |b - c|.\}$$

Hence, we must either have $d_p(a, c) = d_p(a, b)$ or $d_p(a, c) = d_p(b, c)$. ☐

3 The Distinct Distance Problem in $\mathbb{Q}_p^d$

In this section, we prove Theorem 1.1 via induction, showing the minimal number of distinct distances among n points in $\mathbb{Q}_p^d$ is equal to the smallest integer k for which $n \leq p^{kd}$. We reveal a construction which achieves this bound, and finally we invoke the Pigeonhole Principle to establish that no lower value can occur. We begin by recalling the statement of the theorem.

Theorem 3.1 (Theorem 1.1)**.** *The minimal number of distinct distances over all sets of n pairwise distinct points in $\mathbb{Q}_p^d$ is $\lceil \log_p n^{1/d} \rceil$.*

Proof. We prove the theorem by induction on k, the unique integer for which

$$p^{kd} < n \leq p^{(k+1)d}.$$

Base case: For $k = 0$, or equivalently a set with at most p^d points, a construction with one distance is a subset of $\{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d)\}$, where ε_i independently run through $\{0, 1, \dots, p-1\}$. The case where $p = 3$, $d = 2$, and $n = 9$ is shown in Figure 1.

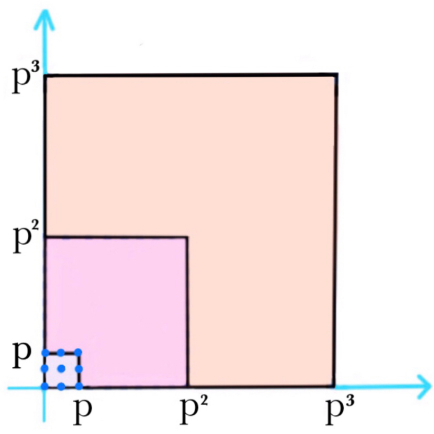

Figure 1: Example construction for $k = 0$.

We will now show that for $p^{kd} < n \leq p^{(k+1)d}$, the minimal number of distinct distances is $k + 1$. A construction for $p^{(k+1)d}$ points with $k + 1$ distinct distances is analogous to the previous (where every coordinate of each point is of the form $x_0 + x_1 p + \cdots + x_k p^k$). We can see an example of this construction for $p = 3$, $d = 2$, and $n = 81$ in Figure 2.

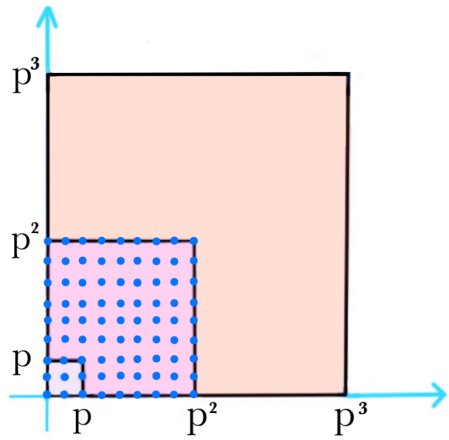

Figure 2: Example construction for $k = 1$.

Now, we will show that for $p^{kd} + 1$ points, there are at least $k + 1$ distances.

Induction step: Assume that $k > 0$, and for $p^{(k-1)d} < n \leq p^{kd}$, the minimal number of distinct distances is k. Let us consider any subset of $p^{kd} + 1$ points with p^{kd} points. We can assume that the number of distinct distances over all subsets with p^{kd} points is k; otherwise, we already have $k + 1$ distances. Let us denote their distances by $p^{-s_1}, \ldots, p^{-s_k}$ where $s_1 < s_2 < \ldots < s_k$. Without loss of generality, we can assume that one of the points is $(0, 0, \ldots, 0)$. The distance between any point and $(0, 0, \ldots, 0)$ must belong to the set $\{p^{-s_1}, \ldots, p^{-s_k}\}$. Thus, at least one coordinate of the nonzero points must have a p^{s_1}, $p^{s_2}, \ldots,$ or p^{s_k} term and no coordinate can contain a p^t term where $t < s_1$.

Each of the points can be written as $a_i = (a_{i1}, a_{i2}, \ldots, a_{id})$. Let us denote the expansion of a_{ij} by $a_{ij} = a_{ij}^{(0)} + a_{ij}^{(1)}p + \cdots$. Then, we assign a_{ij} to the tuple $(a_{ij}^{(s_1)}, a_{ij}^{(s_2)}, \ldots, a_{ij}^{(s_k)})$ for each index j. In doing so, we assign each point a_i to a kd tuple with elements belonging to the set $\{0, 1, \ldots, p-1\}$. Different points must be associated with different tuples because, otherwise, the distance between some points would be different from $p^{-s_1}, p^{-s_2}, \ldots, p^{-s_k}$. Thus, in total there are p^{kd} tuples of coefficients in the $p^{s_1}, p^{s_2}, \ldots, p^{s_k}$ terms.

Let us add another point denoted by $a_{p^{kd}}$. For this point, we can construct a kd-tuple as we did for the previous points. However, the previous points already generated all unique kd-tuples, so this additional point's tuple must coincide with an already existing tuple. Calculating the distance between the new point and one of the old points, we find a distance that is not in $\{p^{-s_1}, p^{-s_2}, \ldots, p^{-s_k}\}$ because all of the p^{s_1}, $p^{s_2}, \ldots, p^{s_k}$ terms will cancel. Hence, there are at least $k + 1$ distances. Since we have a construction with $k + 1$ distances, we have proven the theorem.

□

4 The Unit Distance Problem in $\mathbb{Q}_p^d$

In this section, we prove Theorem 1.2 by partitioning n points in $\mathbb{Q}_p^d$ into p^d subsets based on their coordinate representations, minimizing non-unit distances within subsets to derive the maximum number of unit distances, and then constructing an explicit arrangement of points to achieve this bound. We begin with the statement of the theorem.

Theorem 4.1 (Theorem 1.2)**.** *The maximum number of unit distances over all sets of n pairwise distinct points in $\mathbb{Q}_p^d$ is*

$$\frac{1}{2}(n^2 - n\alpha - \alpha\beta - \beta),$$

where α and β are the unique nonnegative integers such that $\beta < p^d$ and $n = \alpha p^d + \beta$.

Proof. Let $a_i = (a_{i1}, a_{i2}, \dots, a_{id})$ denote a point in $\mathbb{Q}_p^d$, where each $a_{ij} \in \mathbb{Q}_p$. Any coordinate a_{ij} can be written as $a_{ij} = \sum\limits_{v=k}^{+\infty} a_{ij}^{(v)} p^v$, where k is an integer.

Partition n points into p^d subsets as follows: two distinct points a_i and a_j are in the same subset if and only if $(a_{i1}^{(0)}, a_{i2}^{(0)}, \dots, a_{id}^{(0)}) = (a_{j1}^{(0)}, a_{j2}^{(0)}, \dots, a_{jd}^{(0)})$. Any pair of points belonging to the same subset will have a non-unit distance. Denote the number of elements in the i^{th} subset by b_i. Since the subsets are a partition, it follows that $b_1 + b_2 + \cdots + b_{p^d} = n$.

The total number of unit distances for n points is at most

$$\frac{n(n-1)}{2} - \left(\frac{b_1(b_1-1)}{2} + \frac{b_2(b_2-1)}{2} + \cdots + \frac{b_{p^d}(b_{p^d}-1)}{2}\right),$$

which is the total number of distances between pairs minus the number of distances between pairs of points in each subset. To maximize the total number of unit distances, we must minimize the sum

$$b_1(b_1-1) + b_2(b_2-1) + \cdots + b_{p^d}(b_{p^d}-1)$$

under the condition that $b_1 + b_2 + \cdots + b_{p^d} = n$.

Opening the brackets, the expression becomes

$$b_1^2 + b_2^2 + \cdots + b_{p^d}^2 - (b_1 + b_2 + \cdots + b_{p^d}) =$$

$$b_1^2 + b_2^2 + \cdots + b_{p^d}^2 - n.$$

As a result, it is necessary and sufficient to minimize $b_1^2 + b_2^2 + \cdots + b_{p^d}^2$ under the condition.

We will show that if there are two distinct numbers b_i and b_j, decreasing the larger and increasing the smaller by one does not increase the sum $b_1^2+b_2^2+\cdots+b_{p^d}^2$. For example, if $b_i > b_j$, replacing b_i with $b_i' = b_i - 1$ and b_j with $b_j' = b_j + 1$ does not change the total number of points. On the other hand, we know that $b_i^2 + b_j^2 \geq (b_i')^2 + (b_j')^2$ since $b_i - b_j - 1 \geq 0$ and simplifying the right-hand side yields $(b_i - 1)^2 + (b_j + 1)^2 = b_i^2 + b_j^2 - 2(b_i - b_j) + 2 = b_i^2 + b_j^2 - 2(b_i - b_j - 1)$.

Using the replacement of b_i and b_j with b_i' and b_j', respectively, we can achieve the condition that for all i, either $b_i = \alpha$ or $b_i = \alpha + 1$. Since $n = \alpha p^d + \beta$, using the replacement we can achieve $b_1 = b_2 = \cdots = b_\beta = \alpha + 1$ and $b_{\beta+1} = b_{\beta+2} = \cdots = b_{p^d} = \alpha$. In particular, if $\beta = 0$, then all $b_i = \alpha$.

As a result, the number of unit distances does not exceed

$$\begin{aligned}
&\frac{1}{2}\Big(n(n-1) - \big((\alpha+1)\alpha\beta + \alpha(\alpha-1)(p^d-\beta)\big)\Big) \\
&= \frac{1}{2}\big(n^2 - n - (\alpha^2\beta + \alpha\beta + p^d\alpha^2 - p^d\alpha - \alpha^2\beta + \beta\alpha)\big) \\
&= \frac{1}{2}\big(n^2 - \beta - \alpha(\beta + p^d\alpha) - \alpha\beta\big) \\
&= \frac{1}{2}(n^2 - \alpha n - \alpha\beta - \beta).
\end{aligned}$$

To complete the proof, all that remains is to provide a construction for n pairwise distinct points where the number of distinct distances is exactly $\frac{1}{2}(n^2 - n\alpha - \alpha\beta - \beta)$. In order to do this, take all points $(\varepsilon_1, \varepsilon_2, \ldots, e_d) \in \mathbb{Q}_p^d$ where ε_i independently run through the set $\{0, 1, \ldots, p-1\}$ and denote them by $e_1, e_2, \ldots, e_{p^d}$. For $a \in \mathbb{Q}_p$, let $\bar{a} = (a, a, \ldots, a) \in \mathbb{Q}_p^d$. We make a diagram that arranges n points from $\mathbb{Q}_p$, where $n = p^d\alpha + \beta$. We arrange the points into rows and columns so that the distance between any pair in the same column is not unit and distances between all other pairs of points are unit distances. The first α rows each consist of p^d elements and as shown here:

$$\begin{array}{cccc}
e_1 & e_2 & \cdots & e_{p^d} \\
e_1 + \bar{p} & e_2 + \bar{p} & \cdots & e_{p^d} + \bar{p} \\
e_1 + 2\bar{p} & e_2 + 2\bar{p} & \cdots & e_{p^d} + 2\bar{p} \\
 & & \vdots & \\
e_1 + (\alpha-1)\bar{p} & e_2 + (\alpha-1)\bar{p} & \ldots & e_{p^d} + (\alpha-1)\bar{p}.
\end{array}$$

If $\beta > 0$, we add another row with all elements of the form $e_i + \alpha p$ for $1 \leq i \leq \beta$. Thinking about the columns as p^d sets, we can use our previous reasoning to see that the number of unit distances over all pairwise distinct points is $\frac{1}{2}(n^2 - \alpha n - \alpha\beta - \beta)$.

$\square$

5 The Maximal Distinct Distances Problem for Vector Spaces with Non-Archimedean Translation-Invariant Metrics

In this section, we explore the maximal number of distinct distances among n points in a vector space with a non-Archimedean translation-invariant metric. Theorem 1.3 asserts that this number is at most $n-1$, a bound we establish through induction by exploiting the non-Archimedean property. We then build on this in Corollary 1.4, providing a construction in $\mathbb{Q}_p^d$ that achieves exactly $n-1$ distinct distances, confirming the bound's sharpness.

Theorem 5.1. *The number of distinct distances for $n \geq 3$ pairwise distinct points in a vector space equipped with a non-Archimedean translation-invariant metric is at most $n-1$.*

Proof. We will prove the theorem by induction on n.

Base case: For $n = 3$, the theorem statement is satisfied because of the following non-Archimedean property: if $d(a,c) \neq d(b,c)$, then $d(a,b) = \max\{d(a,c), d(b,c)\}$.

Induction step: Suppose that $n \geq 4$ and for all $m < n$ points, the number of distinct distances is at most $m-1$. Let $a_1, \ldots, a_n$ all be pairwise distinct points.

If the distances from a_1 to the other points are pairwise distinct, then we already have $n-1$ distances. Distances between other points in such a set cannot be different from one of the previously mentioned $n-1$ distances because for all $2 \leq i, j \leq n$,

$$|a_i - a_j| = |(a_i - a_1) - (a_j - a_1)| = \max\{|a_i - a_1|, |a_j - a_1|\},$$

where $|\cdot|$ denotes a non-Archimedean norm.

We will now examine the case where not all distances $|a_i - a_1|$ with $i \geq 2$ are distinct. We will partition the set of differences $a_i - a_1$ into subsets based on the value of $|a_i - a_1|$. We can change the notation so that the partition has the following form:

$$a_2^{(1)} - a_1, a_3^{(1)} - a_1, \ldots, a_{r_1+1}^{(1)} - a_1,$$

$$a_2^{(2)} - a_1, a_3^{(2)} - a_1, \ldots, a_{r_2+1}^{(2)} - a_1,$$

$$\vdots$$

$$a_2^{(s)} - a_1, a_3^{(s)} - a_1, \ldots, a_{r_s+1}^{(s)} - a_1,$$

where r_i is the number of elements in the i^{th} subset and s is the number of subsets. Without loss of generality, we can assume that the subsets are organized by the defining distance for each subset in decreasing order. It is clear that $r_1 + r_2 + \cdots + r_s = n - 1$.

Let $b_j^{(i)} = a_j^{(i)} - a_1$ where $2 \leq j \leq r_i + 1$. Since $r_i \leq n - 1 < n$, by the induction hypothesis, there are at most $r_i - 1$ distinct distances between points $b_j^{(i)}$. However, the distances between points $b_j^{(i)}$ are the same as the distances between the corresponding points $a_j^{(i)}$ since $|b_{j_2}^{(i)} - b_{j_1}^{(i)}| = |a_{j_2}^{(i)} - a_{j_1}^{(i)}|$ for all $2 \leq j_1, j_2 \leq r_i + 1$ and $1 \leq i \leq s$. Thus, the differences $a_{j_2}^{(i)} - a_{j_1}^{(i)}$ and $a_2^{(i)} - a_1$ give at most $r_1 + r_2 + \ldots + r_s = n - 1$ distinct distances.

Finding distances between other pairs of points cannot yield a distance that is different from one of the previously constructed distances because of the following: Take points $a_{j_1}^{(i_1)}$ and $a_{j_2}^{(i_2)}$ where $i_1 \neq i_2$. If $i_1 > i_2$,

$$|a_{j_1}^{(i_1)} - a_{j_2}^{(i_2)}| = |(a_{j_1}^{(i_1)} - a_1) - (a_{j_2}^{(i_2)} - a_1)| = \max\{|a_{j_1}^{(j_1)} - a_1|, |a_{j_2}^{(i_2)} - a_2|\} = |a_{j_1}^{(i_1)} - a_1|.$$

We have already accounted for the right side of the equation above to reach the $n-1$ previously constructed points. Therefore, $|a_{j_1}^{(i_1)} - a_{j_2}^{(i_2)}|$ does not yield additional distances, which completes the proof.

□

Corollary 5.2. *The maximum number of distinct distances for n pairwise distinct points in $\mathbb{Q}_p^d$ is $n - 1$.*

Proof. As a result of the previous theorem, the maximum number of distances in $\mathbb{Q}_p^d$ is at most $n - 1$. By providing a construction for n points in $\mathbb{Q}_p^d$ that yields exactly $n - 1$ distances, we prove the corollary. For the construction, take the following n points:

$$(1, 0, 0, \ldots, 0)$$
$$(p, 0, 0, \ldots, 0)$$
$$(p^2, 0, 0, \ldots, 0)$$
$$\vdots$$
$$(p^{n-1}, 0, 0, \ldots, 0).$$

□

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